Minimum Weight Design of Stiffened Fiber Composite Cylinders

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A structural synthesis capability for stiffened fiber composite cylindrical shells has been developed. The design variables are the configuration and material parameters. The material parameters considered are the fiber volume content and the ply orientations in the skin. Both longitudinal and circumferential stiffeners are assumed to have hat cross sections. The instability loads of the heterogeneous anisotropic cylindrical shell under combined axial, radial, and torsional loading are calculated with a smeared stiffener theory. In the synthesis scheme, multiple load conditions on the cylinder are permissible. The optimal design problem is cast into a model of a nonlinear mathematical programming problem which is solved with the penalty function technique of Fiacco and McCormick. Since weight is independent of the ply orientations there exist alternative optima. Special modifications on the iteration procedure are made, so that the ply angles will change in such a way that the over-all strength of the cylinder is increased. Several numerical examples are discussed.

Nomenclature

CSB	= circumferential stiffener buckling
CSF	= circumferential stiffener material failure
d_x, d_θ	= depth of stiffeners
GB	= gross buckling
h_x, h_θ	= width of stiffeners
I	= number of ply in skin
Init	= initial
LB, L_m	= lower bound
LSB	= longitudinal stiffener buckling
LSF	= longitudinal stiffener material failure
l, l_x, l_θ	= length of cylinder, distance between stiffeners
$\bar{N}_x$	= applied axial force
PB	= panel buckling
p, q	= radial pressure
R	= radius
r	= constant multiplier
SB	= skin buckling
SF	= skin material failure
s	= moving directions
T	= torsional force
t	= thickness
UB, U_m	= upper bound
v_s	= fiber volume content
x	= set of design variables
α	= ply angle, load ratio
β	= load ratio
γ	= load ratio, weight density
δ	= small positive number, term controlling factor
θ	= ply angles, θ axis
λ	= modification factor
ϕ	= applied loads $\phi_1 = \bar{N}_x$, $\phi_2 = Rq$, $\phi_3 = M/2R^2$

Subscripts

c	= composite
cr	= critical, index
f	= fiber
i	= index, imaginary part
j	= index
l	= fiber direction
LT	= LT plane
m	= matrix, material, index
P	= index
Q	= number of load conditions
q	= index
s	= stiffener, skin, structure
T	= transverse direction of unidirectional ply
x	= x-axis direction or longitudinal direction
θ	= circumferential direction

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Introduction

WITH the development of new composite materials, a design engineer has the opportunity of ultimately being able to design a structural configuration and material simultaneously. However, to accomplish this task it will require fundamental advances in structural synthesis as well as an improved understanding of the structural behavior of composite structures. This work is aimed at advancing the techniques of structural synthesis in design with fibrous composites. By extending the technique to composition and material variables, it provides a tool to assist the engineer in the intelligent exploitation of the potential offered by composite materials.

The basic concepts of structural synthesis were developed and reported by Schmit and his associates.[1,2] The concepts have been applied successfully to the optimum design of integrally stiffened cylindrical shells by Kicher[3] and Schmit-Morrow-Kicher.[4] In general, structural synthesis, or automated optimum design of structural systems, can be cast into a model of a mathematical programming problem which seeks to maximize or minimize a function of several variables. The variables are subjected to certain constraints which are not amenable to solution by the classical methods of calculus. Composite structures must be designed with an integrated design-analysis approach supported closely by materials and processing criteria. In order to carry out this work successfully, a designer is required to have a comprehensive knowledge of design, manufacturing, and analytical methods. This complicates greatly the design processes. It is hoped, in this respect, that an extended technique of structural synthesis will assist an engineer to achieve an optimum structural design.

The main features of the computer capability developed for finding a minimum weight design of stiffened fiber composite cylinders can be summarized in the following: 1) Multiple load conditions—each load condition composed of axial, radial, and torsional load combinations. 2) Design begins with the elastic properties of the constituent materials, fiber and matrix. 3) Design variables include the material parameters, fiber volume content and fiber orientations. 4) The buckling analysis of the heterogeneous anisotropic cylindrical shell is based upon the small deflection theory of thin shells. The cylinder is assumed to buckle into a torsional wave form. 5) Eight sets of boundary conditions are provided. 6) Bounds on all design variables are permitted. 7) Optimum design is obtained by minimizing the objective function with the design variables subject to side and behavior constraints.

Figure 1 shows a sketch of the shell with stiffeners located inside the cylinder wall. The design variables are the configuration parameters: dimensions and spacings of the stiffeners; and the material parameters: fiber orientations and fiber content. The thickness of the composite is determined from the fiber volume content. The stiffened shell is designed against the following modes of failure: 1) gross buckling, 2) panel buckling, 3) skin buckling, 4) stiffener bucklings, 5) combined-stress failure in a ply, 6) delamination failure between two adjacent plies. Behavior constraints are formulated based on the failure stresses and strains. Side constraints are the lower and upper bounds on the design variables, and the limits on the geometric admissibilities among the dimensions and spacings of the stiffeners. Together they form the constraint set. The objective function is the total weight of the cylinder.

The stiffeners in both the longitudinal and circumferential directions are assumed to be closely spaced, then the elastic properties of the stiffeners may be considered to be smearable in the two directions over the skin. Thus, the force-displacement relations are obtained by performing the appropriate integrations of the stresses over the skin and the stiffeners with the mid-surface of the skin taken as the reference surface. Timenshenko's buckling equilibrium equations[5] are used for the buckling analysis of the cylinder. The skin buckling loads are approximated with Donnell's equations.[6] The hat section stiffeners are analyzed as a collection of plate elements. Methods to determine the limiting strength of composite materials are based on the work of Chamis.[7] Material failure modes considered are the individual ply failures and the ply delaminations between two adjacent plies.

The minimum weight design problem is cast into a model of a nonlinear mathematical programming problem. The Fiacco-McCormick penalty function technique is used for solving the programming problem. Since the objective or weight function is independent of the ply orientations, special modifications have been made in the synthesis scheme so that the weight is minimized with respect to all design variables and the strength of the structure is maximized with respect to the ply angles.

Problem Formulation

The design variables are the configuration parameters, dimensions and spacings of the stiffeners, and the material parameters, fiber volume content and fiber orientations. Figure 1 shows the appropriate configuration parameters. Let the design variables be represented by a vector $\mathbf{x}$, i.e.,

$$\mathbf{x} = (d_x, d_\theta, h_x, h_\theta, l_x, l_\theta, v_s, \alpha_1, \alpha_2, \ldots, \alpha_I) \quad (1)$$

Then any particular design is defined by a point in the design space located by the corresponding vector $\mathbf{x}$. The subscript I in Eq. (1) denotes the number of plies in the skin. With the total number of plies specified, the thickness of the cylinder wall is determined from the fiber volume content v_s.

The constraints which set the upper and lower bounds on

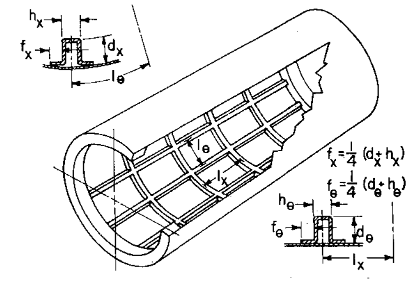


Fig. 1 Integrally stiffened cylinder with hat cross-section stiffeners.

the variables are defined as follows:

$$g_m(\mathbf{x}) = (U_m - x_m)/(U_m - L_m) \geq 0, \quad m = 1,2, \ldots, (7 + I)$$
$$g_m(\mathbf{x}) = (x_m - L_m)/(U_m - L_m) \geq 0, \quad m = (8 + I), (9 + I), \ldots, 2(7 + I) \quad (2)$$

where U_m is the upper bound and L_m is the lower bound on the mth design variable x_m. To prevent the stiffeners from overlapping each other, two geometric compatibility relations are formed among the dimensions and spacings of the stiffeners. The relations are

$$g_m(\mathbf{x}) = (l_\theta - \tfrac{3}{2}h_x + \tfrac{1}{2}d_x)/l_\theta \geq 0, \; m = 2(7 + I) + 1$$
and
$$g_m(\mathbf{x}) = (l_x - \tfrac{3}{2}h_\theta - \tfrac{1}{2}d_\theta)/l_x \geq 0, \; m = 2(7 + I) + 2 \quad (3)$$

thus, there are $(16 + 2I)$ side constraints in all.

The behavior constraints are formulated based on the failure stresses and strains. Specific failure modes considered are 1) gross buckling—buckling of the entire cylinder, 2) panel buckling—buckling of the cylinder between two adjacent rings, 3) skin buckling—buckling of the cylindrical skin element, 4) longitudinal stiffener buckling, 5) circumferential stiffener buckling, 6) material failure—skin, 7) material failure—longitudinal stiffeners, 8) material failure—circumferential stiffeners.

All of the failure modes are assumed to be independent of one another. When the stiffened shell fails in one of the modes it is considered unacceptable. Since the cylinder is designed to be subjected to a multiplicity of Q distinct load conditions, there are a total of $(8Q)$ failure modes to be considered simultaneously. Let $f_{pq}(x)$ denote the value of the pth behavior parameter in the qth load condition and let $[f_{pq}(x)]_{cr}$ denote the corresponding critical or limiting value. Then the behavior constraints may be expressed in the following form:

$$g_{pq}(\mathbf{x}) = 1 - \frac{f_{pq}(\mathbf{x})}{[f_{pq}(\mathbf{x})]_{cr}} \geq 0, \quad p = 1, \ldots, 8; \; q = 1, \ldots, Q \quad (4)$$

where p and q identify the pth failure mode and the qth load condition, respectively. To simplify the notation, the behavior constraints can be rewritten as follows

$$g_m(\mathbf{x}) = 1 - \frac{f_m(\mathbf{x})}{(f_m)_{cr}} \geq 0$$
$$m = (16 + 2I) + p + (q - 1)Q \quad (5)$$
$$m = (16 + 2I) + 1, \ldots, (16 + 2I) + 8Q$$

where $(16 + 2I)$ is the number of side constraints and Q is the number of distinct load conditions. The behavior parameters $[f_{pq}(\mathbf{x})]$ which represent the buckling loads and limiting stresses of a stiffened cylinder are obtained from the analysis of fiber composite shells outlined in the next section.

The object function, or weight function, of the cylinder is defined in terms of the design variables and the over-all dimen-

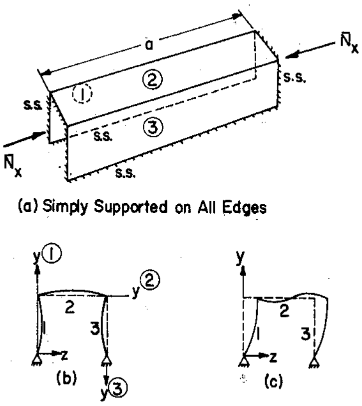


Fig. 2 Buckling modes of an idealized hat cross-section stiffener.

sions of the cylinder. The expression for the total weight of the stiffened cylindrical shell structure is

$$w = \pi l(a^2 - b^2)[\gamma_f v_s + \gamma_m(1 - v_s)] + n_x\{[2(f_x + d_x) + (h_x - 2t_x)]t_x(l - \delta_x n_\theta h_\theta)\}[\gamma_f v_x + \gamma_m(1 - v_x)] + n_\theta(\pi^*\{(d^2 - (d \mp t_\theta)^2)(2f_\theta) + (d^2 - (d \mp d_\theta)^2)(2t_\theta) + [(d \mp d_\theta \pm t_\theta)^2 - (d \mp d_\theta)^2](h_\theta - 2t_\theta)\} - \delta_\theta[2(f_\theta + d_\theta) + (h_\theta - 2t_\theta)]\cdot(t_\theta n_x h_x))[\gamma_f v + \gamma_m(1 - v_\theta)] \quad (6)$$

where

$$a = R + \frac{t_s}{2}, b = T - \frac{t_s}{2}, c = R \pm \frac{t_s}{2}, d = R \pm \frac{t_s}{2}$$

$$f_x = \tfrac{1}{4}(d_x + h_x), f_\theta = \tfrac{1}{4}(d_\theta + h_\theta)$$

$$\gamma_f = \text{weight density of fiber, lb/in.}^3$$

$$\gamma_m = \text{weight density of matrix, lb/in.}^3$$

$$n_x = \frac{2\pi R}{l_\theta}\begin{cases}\text{number of longitudinal stiffeners} \\ \text{(assuming } n_\theta \text{ to be fraction)}\end{cases}$$

$$\delta_x = \begin{cases}0 \text{ for continuous longitudinal stiffeners} \\ 1 \text{ for discontinuous longitudinal stiffeners}\end{cases}$$

$$\pi^* = \begin{cases}+\pi \text{ circumferential stiffeners are outside} \\ -\pi \text{ circumferential stiffeners are outside}\end{cases}$$

$\pm$ use upper sign for inside circumferential stiffeners
or
$\mp$ use lower sign for outside circumferential stiffeners

$$t_k = \tfrac{1}{2}I_k d_f(N_f\pi/v_k)^{1/2}, \text{ composite thickness}^7$$

where $k = s$, x, and θ for skin, longitudinal stiffener, and circumferential stiffener, respectively, I = number of plies, d_f = diameter of fiber, N_f = number of fibers per roving end, v = percent fiber volume content. By taking n_x and n_θ as fractions instead of integers, the weight function [Eq. (6)] becomes a continuous function with respect to all design variables. This approximation avoids the difficult task of solving an integer programming problem. It is assumed however that the values n_x and n_θ are sufficiently large that rounding off has a negligible effect. This approximation is consistent with the smearable stiffener assumptions used in the gross behavior analysis.

The problem may now be stated as follows: find a set of design variables $\mathbf{x}$ such that the weight function [Eq. (6)] is at a minimum with the design variables subject to a set of constraints [Eqs. (2, 3, and 5)].

Analysis of Composite Shells

To calculate the behavior parameters f_{pq} and $(f_{pq})_{cr}$, a complete analysis of the shell structure has been formulated and the details can be found in Ref. 8.

The linear eigenvalue analysis for gross and panel buckling is based on a method similar to that presented by Cheng and Ho.[9]

In searching for the buckling loads of a cylindrical shell under combined loadings, it is assumed that the load ratios among the applied axial load, radial pressure, and torsional force remain constant. A change of one load will automatically change the other two loads according to the fixed ratios. Let ϕ_1, ϕ_2, and ϕ_3, be the applied axial load, radial load, and torsional force, respectively, then the applied primary load is defined as the first nonzero load of ϕ_1, ϕ_2, and ϕ_3, i.e., P = (1st nonzero load of ϕ_1, ϕ_2, and ϕ_3. The load ratio parameters are defined as:

$$\alpha = \phi_1/P, \beta = \phi_2/P, \gamma = \phi_3/P \quad (7)$$

With the applied load ratios known, the forces on the cylinder corresponding to any primary load P are

$$\phi_1 = \alpha P, \phi_2 = \beta P, \phi_3 = \gamma P \quad (8)$$

Thus, the critical value of P is considered as the primary buckling load and is actually obtained through the buckling analysis of Ref. 8.

The behavior parameters f_{pq} for the gross and panel buckling modes can be expressed as follows: $f_{pq} = P_q$, $p = 1,2$; $q = 1,2, \ldots, Q$ where p denotes the pth failure mode, q denotes the qth load condition and P is the applied primary load. The corresponding $(f_{pq})_{cr}$ are defined as $(f_{pq})_{cr} = (P_{pq})_{cr}$, $p = 1,2$; $q = 1,2, \ldots, Q$ where $(P_{pq})_{cr}$ is obtained from the buckling analysis.

For the balance of the failure modes the local applied forces are determined through a prebuckle analysis. Thus, $f_{pq} = F_{pq}$, $p = 3,4, \ldots, 8$; $q = 1,2, \ldots, Q$ where F_{pq} is the local applied force for the pth failure mode and qth load condition. The corresponding $(f_{pq})_{cr}$ are defined in the following: $(f_{pq})_{cr} = (F_{pq})_{cr}$, $p = 3,4, \ldots, 8$; $q = 1,2, \ldots, Q$. For skin buckling, $(F_{3q})_{cr}$ are obtained from the analysis of skin buckling based on Donnell type equations, for orthotropic skin with reduced flexural stiffnesses.[10] The circumferential stiffeners are assumed to be straight between two longitudinal stiffeners, since the spacings are assumed close enough to smear out the elastic properties over the skin. Therefore the buckling forces $(f_{5q})_{cr}$ are approximated with the same stiffener analysis. The hat cross-section stiffener is idealized as a three-member plate structure, simply supported along the two joints with the skin as shown in Fig. 2. In calculating the stiffener buckling stresses it is assumed that the top corners of the idealized structure can displace laterally, i.e., side-sway is permissible.

The parameters $(f_{6q})_{cr}$, $(f_{7q})_{cr}$, and $(f_{8q})_{cr}$ for the skin, longitudinal stiffener and circumferential stiffener material failures, respectively, are determined through the material failure criteria of Ref. 7.

Method of Synthesis

The synthesis scheme employs the penalty function technique of Fiacco-McCormick.[11] This technique requires a transformation of the constrained problem into a sequence of unconstrained minimization problems. Since the weight function is independent of the ply orientations, special modifications on the synthesis scheme are made. Thus, the weight is minimized with respect to all the design variables and the strength of structure is maximized with respect to the ply angles.

To convert the constrained minimization problem to a sequence of unconstrained minimization problems, the following unconstrained function is formed[11]:

$$F(\mathbf{x},r) = w(\mathbf{x}) + r\sum(1/g_m)(\mathbf{x}) \quad (9)$$

where r is a constant multiplier and $r > 0$. Let $r = r_1$ and $\mathbf{x}_0$ be strictly inside the constraint set, then $F(\mathbf{x},r_1)$ can be minimized starting from $\mathbf{x}_0$ subject to no constraints. Repeating this minimization process for a sequence of r values $r_1 > r_2 > \ldots r_k > 0$, Fiacco and McCormick[11] have shown that for a convex function as r approaches zero the optimal solution to the unconstrained problem approaches the optimal solution to the constrained problem, i.e., $F^*(\mathbf{x},r_k) \rightarrow w^*(\mathbf{x})$.

1. Minimization of the Unconstrained Function

The unconstrained function $F(\mathbf{x},r)$ may be rewritten as follows in order to facilitate the computations for the components of the gradient.

$$f(\mathbf{x},r) = w(\mathbf{x}) + r[S(\mathbf{x}) + B(\mathbf{x})] \tag{10}$$

where

$$S(\mathbf{x}) = \sum_m \frac{1}{g_m(\mathbf{x})} \quad m = 1, \ldots, (16 + 2I)$$

$$B(\mathbf{x}) = \sum_m \frac{1}{g_m(\mathbf{x})} \quad m = (16 + 2I) + 1, \ldots, (16 + 2I) + 8Q \tag{11}$$

The term $S(\mathbf{x})$ represents the contribution of the side constraints to the penalty function in Eq. (9) and the term $B(\mathbf{x})$ represents the contribution of the behavior constraints to the penalty function. The gradient of the unconstrained function is:

$$\nabla F = \nabla w + r[\nabla S + \nabla B] \tag{12}$$

∇w and ∇S can be obtained explicitly by partial differentiation with respect to each of the design variables.

The gradient to $B(\mathbf{x})$, i.e., to behavior constraint part of the penalty function, is determined using a finite difference method. Its components can be expressed as follows:

$$\partial B/\partial x_m = [B(x_m) - B(\mathbf{x})]/\Delta x_m \quad m = 1, \ldots, (7 + I) \tag{13}$$

where $x_m = (x_m + \Delta x_m), \ldots, x_{7+I}$. In order to obtain a numerical value for function $B(\mathbf{x})$, it is necessary to frequently repeat the analysis of fiber composite shells. However, in calculating $B(\mathbf{x}_m)$ an analysis is performed by using the same critical mode shapes as obtained for the design point $\mathbf{x}$.[3]

2. Modified Gradient and Moving Directions

The unconstrained function Eq. (9) is minimized with the Fletcher-Powell's method[12] for each given $r = r_i > 0$. After repeating this process for a number of r values $r_1 > r_2 > \ldots r_k > 0$, the weight function $w(\mathbf{x})$ dominates the calculation of the unconstrained function $F(\mathbf{x},r)$. Further, by reducing r the influence of the term $r\Sigma 1/g_i$, which penalizes closeness to the constraint boundaries, is reduced and, in minimizing $F(\mathbf{x},r)$, more effort is concentrated upon reducing $w(\mathbf{x})$. Since $w(\mathbf{x})$ is independent of α_i's, this effort has little effect on ply angles. Theoretically the Fiacco-McCormick method will seek out the optimum but the decreasing sensitivity of the ply angles requires long and expensive computation. In an attempt to speed the convergence, the influence of the ply angles on the penalty function is artifically inflated.

The gradients to $B(\mathbf{x})$ can be thought of as having two components. One component represents the partial derivatives to the sizing variables while the other component arises from the partial derivatives to the ply angles. Because of the nature of the Fiacco-McCormick penalty function technique the total sizing variable derivatives $\partial F/\partial x_i$, $1 = 1,2, \ldots, 7$, are substantially larger than the ply angle derivatives $r[(\partial S/\partial \alpha_i) + (\partial B/\partial \alpha_i)]$, when r becomes very small or when $\mathbf{x}$ is not very close to the constraints. Thus, a move in the direction of ∇F will only change the ply angles α_i's slightly. If two values of r were available, one for the sizing variables and one for the ply angles, a more efficient minimization algorithm might result. To accomplish this task a "modified gradient" is constructed from Eq. (12). Thus,

$$\nabla F^* = \nabla W + r\nabla S + r[\lambda]\nabla B \tag{14}$$

where $[\lambda]$ is a modification matrix and it is defined by the following expression:

$$\nabla F^* = \begin{Bmatrix} \dfrac{\partial w}{\partial x_1} \\ \cdot \\ \cdot \\ \dfrac{\partial w}{\partial x_7} \\ \cdots \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} + r\nabla S + r \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdot & \cdot & 0 & & & & & \\ 0 & 1 & 0 & \cdot & 0 & & & & & \\ \cdot & & & & & & & 0 & & \\ \cdot & & & & & & & & & \\ 0 & \cdot & \cdot & \cdot & 1 & & & & & \\ \hline & & & & & \frac{\lambda}{r} & 0 & \cdot & \cdot & 0 \\ & & 0 & & & 0 & \frac{\lambda}{r} & 0 & \cdot & 0 \\ & & & & & 0 & & & & \\ & & & & & \cdot & & & & \\ & & & & & 0 & \cdot & \cdot & \cdot & \frac{\lambda}{r} \end{array}\right] \begin{Bmatrix} \dfrac{\partial B}{\partial x_1} \\ \cdot \\ \cdot \\ \dfrac{\partial B}{\partial x_7} \\ \dfrac{\partial B}{\partial \alpha_1} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial B}{\partial \alpha_I} \end{Bmatrix} \tag{15}$$

If $\lambda = r$, this expression is exactly the gradient equation [Eq. (12)] i.e., no modification. If $\lambda = 10r$, it implies that the penalty on the partial derivatives of $B(\mathbf{x})$ with respect to ply angles α_i's is ten times more severe than on other partial derivatives. Or if $\lambda = 1$, this means that $(\partial B/\partial \alpha_i)$'s are weighed numerically in the same scale as the partial derivatives of $w(\mathbf{x})$ with respect to the design variables. Therefore the modification matrix $[\lambda]$ provides a tool which can be used to emphasize certain partial derivatives for the behavior constraint part of the penalty function. Numerical examples of the previous cases are given in the next section.

The unconstrained minimization problem is solved with the Fletcher-Powell method.[12] A single iteration step of the method is described by the following: Let H_0 = any positive definite matrix, e.g., the identity matrix and $s_i = -H_i(\nabla F)_i^*$ moving direction. Choose $a = a_i$ by minimizing $F(\mathbf{x}_i + a\mathbf{s}_i,r)$. Let $\sigma_i = a_i\mathbf{s}_i$, then, $\mathbf{x}_{i+1} = \mathbf{x}_i + \boldsymbol{\sigma}_i$ and $H_{i+1} = H_i + A_i + B_i$ where $A_i = \boldsymbol{\sigma}_i\boldsymbol{\sigma}_i^T/\boldsymbol{\sigma}_i^T\mathbf{y}_i$ and $B_i = -H_i\mathbf{y}_i\mathbf{y}_i^T/\mathbf{y}_i^TH_i\mathbf{y}_i$ and $y_i = (\nabla F)_{i+1}^* - (\nabla F)_i^*$.

3. One-Dimensional Minimization

In applying the Fletcher-Powell method to the unconstrained minimization it is required to find the minimum value of F along each move direction $\mathbf{s}_i$, i.e., to find $a = a_i$ by minimizing $F(\mathbf{x}_i + a\mathbf{s}_i,r)$. The quantity $F(a)$ is assumed to be a unimodal function which implies, roughly speaking, that it has only one minimum. Since the derivatives, $\partial F/\partial x_i$, are difficult to compute, a quadratic interpolation is used in the one-dimensional minimization. A procedure of the quadratic interpolation may be outlined as follows.

Stage 1 The $\mathbf{s}$ vector is normalized by dividing each component of $\mathbf{s}$ with $\Delta = \max_j|\mathbf{s}_j|$ in x.

Stage 2 $F(a)$ is evaluated at the points $a = 0,1,2,4, \ldots, \alpha,\beta,\gamma$ where γ is the first of these values at which F has increased. If $F(1) > 10F(0)$ then the components of $\mathbf{s}$ are divided by a factor of 3 and the step is repeated.

Table 1 Design data for cylindrical shell #1 for various values of λ

	d_x	d_θ	h_x	h_θ	l_x	l_θ	v_s	α_1	α_2	Weight	t_s
$\lambda = r$	0.142	0.269	0.160	0.275	12.0	4.96	0.687*	72.6	−72.9	26.7	0.0218
$\lambda = 10r$	0.107[a]	0.195	0.107[a]	0.426	10.2	3.60	0.697*	69.7	−69.7	26.8	0.0216
$\lambda = 1$	0.125	0.235	0.160	0.387	13.7	7.76	0.698*	84.7	−83.6	25.0	0.0216
Initial	0.500	1.000	0.500	0.500	10.0	2.00	0.500	45.0	−45.0	61.5	0.0255

				Load/critical load				
	GB	PB	SB	LSB	CSB	SF	LSF	CSF
$\lambda = r$	...	...	0.987	0.034	...	0.141	0.508	0.034
$\lambda = 10r$	...	...	0.976	0.016	...	0.198	0.527	0.047
$\lambda = 1$	...	...	0.925	0.051	...	0.122	0.763	0.011

[a] Active side constraint; radius = 30 in.; length = 100 in.; load: $N_x = -100$ lb/in.; LS 3-ply (5°, 0°, −5°), $v_x = 0.6$; CS 2-ply (±10°), $v_\theta = 0.6$; skin 2-ply; material Thornel 50.

Table 2 Design data for cylindrical shell #2, λ = 1 (4 cycles)[a]

						Design variables							
	d_x	d_θ	h_x	h_θ	l_x	l_θ	v_s	α_1	α_2	α_3	α_4	Weight	t_s
Final	0.513	1.55	0.422	0.511	9.52	2.39	0.692	110	19.5	−19.8	−108	15.9	0.0434
Initial	0.800	1.60	0.600	0.600	9.50	2.23	0.600	90	15.0	−50.0	−90	20.1	0.0467
UB	1.50	3.00	0.800	0.800	24.0	8.00	0.750	180	90.0	90.0	0		
LB	0.200	0.200	0.200	0.200	1.00	0.800	0.300	0	−90.0	−90.0	−180		

					Load/critical load						
	GB	PB	SB	LSB	CSB	SF	LSF	CSF	N_x	P	T
Final	0.075 (2)	0.937 (6)	0.690 (6,1)	0.241	...	0.365	0.915	0.033	−1800	0	0
Initial	0.079	0.580	0.021	0.416	...	0.314	0.648	0.001			

[a] Radius 11 in.; length 48 in.; L stiffener 4-p (±10°, ∓ 10°); $v_x = 0.60$; C stiffener 4-p (±10°, ∓10°); $v_\theta = 0.60$; skin 4-p; material Thornel-40.

Table 3 Design data for cylindrical shell #3 for various load conditions[a]

	d_x	d_θ	h_x	h_θ	l_x	l_θ	v_s	α_1	α_2	α_3	Weight	t_s
LC #1	0.119	0.125	0.196	0.305	6.20	3.44	0.740[b]	51.4	−59.8	77.4	36.8	0.0315
LC #2	0.130	0.329	0.622	0.559	6.10	3.45	0.713	72.0	−81.8	71.9	42.3	0.0320
LC #3	0.375	0.670	0.471	0.698	6.02	3.06	0.512	41.2	−42.1	41.1	55.1	0.0378
Multiple loads	0.239	0.572	0.470	0.670	6.02	3.06	0.54	57.0	−59.5	75.1	50.4	0.0368
Multiple loads	0.148	0.470	0.386	0.627	6.05	3.12	0.630	69.5	−57.9	75.6	44.3	0.0341

	Load	GB	PB	SB	LSB	CSB	SF	LSF	CSF
LC #1	$N_x = -200$	0.435	0.848	0.708	0.169	...	0.502	0.731	0.096
LC #2	$p = 4$ psi	0.938	0.023	0.950	...	0.142	0.672	0.010	0.357
LC #3	$T = 566$	0.450	0.313	0.625	...	...	0.962	0.095	0.096
Multiple loads	$N_x = -200$	0.145	0.725	0.279	0.586	...	0.134	0.441	0.048
	$p = 4$	0.284	0.030	0.752	...	0.242	0.128	0.025	0.430
	$T = 566$	0.534	0.314	0.490	...	...	0.970	0.006	0.128
Multiple loads	$N = -200$	0.260	0.806	0.350	0.557	...	Skin failure mode	0.617	0.058
	$p = 4$	0.574	0.301	0.816	...	0.192	suppressed	0.030	0.393
	$T = 566$	0.851	0.958	0.274	0.322	...		0.192	0.149

[a] Radius 30 in.; length 100 in.; L stiffener 2-p (5°, −5°); $v_x = 0.60$; C stiffener 2-p (16°, −16°); $v_\theta = 0.60$; skin 3-p; material Thornel-40; boundary conditions: $w = 0$; $M_x = 0$; $N_{x\theta} + (M_{x\theta}/R) + N_x(\partial v/\partial x) = 0$; $N_x + (T/R)u_{,\theta} = 0$.
[b] Active side constraint.

Table 4 Design data for cylindrical shell #4[a]

					Design variables					
	d_x	d_θ	h_x	h_θ	l_x	l_θ	v_s[b]	1	Weight	t_s*
Final	0.382	2.70	0.558	0.270	5.64	1.71	0.133	0	608	0.027
Initial	0.600	2.75	0.600	2.75	5.60	2.00	0.001	0	2236	0.272
UB	2.00	3.00	2.00	3.00	20.0	10.0	0.950	90		
LB	0.150	0.200	0.200	0.200	1.50	0.400	0.0007	−90		

					Load/critical load						
	GB	PBS	SB	LSB	CSB	SF	LSF	CSF	N_x	P	T
Final	0.952	0.756	0.908	0.070	...	0.929	0.650	0.103	−2820	1.2	0

[a] Radius 60 in.; length 165 in.; L stiffener $t_x = 0.0625$; C stiffener $t_\theta = 0.0100$; skin isotropic; material aluminum; boundary conditions: $w = 0$; $M_x = 0$; $N_x = 0$; $v = 0$.
[b] $t_s = f(v_s)$.

Table 5 Mechanical properties for isotropic and unidirectional ply material

Material	E_L, psi	E_T, psi	μ_{LT}	G_{LT}, psi
Aluminum	10×10^6	10×10^6	0.333	3.89×10^6
Matrix (ERL-2256)[a]	0.57×10^6	0.57×10^6	0.360	0.209×10^6
"Thornel" 50[a] ($\mu_{TR} = 0.2$)	45×10^6	1.5×10^6	0.250	1.5×10^6
"Thornel" 40* ($\mu_{TR} = 0.2$)	40×10^6	1.5×10^6	0.250	1.5×10^6

[a] A product of Union Carbide Corporation.

Stage 3 A quadratic polynomial is now fitted to the three values $F(\alpha)$, $F(\beta)$, and $F(\gamma)$. Its minimum occurs at $a = a_c$ where a_c may be expressed as follows:

$$a_c = \frac{1}{2}\,\frac{F(\alpha)(\gamma^2 - \beta^2) + F(\beta)(\alpha^2 - \gamma^2) + F(\gamma)(\beta^2 - \alpha^2)}{F(\alpha)(\gamma - \beta) + F(\beta)(\alpha - \gamma) + F(\gamma)(\beta - \alpha)}$$

If $F(a_c) \leq F(\beta)$ then a_3 is accepted as the estimate of a_i. Otherwise β is taken as a_i.

4. Stop Criteria

Convergence criteria are required to terminate: 1) a cycle of the Fletcher-Powell unconstrained minimization and 2) the over-all minimization. Two convergence criteria are provided for each of the previous cases.

A cycle of the unconstrained minimization is considered converged if

$$1)\quad \left|\frac{\partial F}{\partial x_j}\right| < \epsilon \quad j = 1,2, \ldots, n$$

or

$$2)\quad \left|\frac{F(\mathbf{x}_{i-1}) - F(\mathbf{x}_i)}{F(\mathbf{x}_i)}\right| < \delta \text{ and } \left|\frac{F(\mathbf{x}_{i-2}) - F(\mathbf{x}_i)}{F(\mathbf{x}_i)}\right| < \delta$$

where $\epsilon = 0.01$ and $\delta = 0.006$. Criterion 1 is seldom reached in actual computations despite the fact that it works well for simple quadratic functions. Criterion 2 is more dependable, provided that it is satisfied for at least two successive values of i.

The over-all program is terminated when the difference between upper and lower bounds of the Fiacco-McCormick primal-dual method becomes less than some given tolerance. The criterion may be expressed in the following form: $(F - G)/F < \epsilon$ where F is the unconstrained objective function which gives the upper bound. G is the dual objective function which gives the lower bound, and $G(\mathbf{x},r) = w(\mathbf{x}) - r\Sigma_m 1/g_m(\mathbf{x})$. It differs from the primal objective function $F(\mathbf{x},r)$ by the negative sign on the penalty term. This criterion has been proven to converge to the solution for a strictly convex function $F(\mathbf{x},r)$. For others it may not converge, thus, the program will also be terminated by a preassigned number of cycles. This second criterion provides a means for stopping the program if the convergence is too slow.

Results and Discussions

Numerical results presented in this section focus on three subjects: 1) to illustrate the reasons and justifications for the modification of the gradient ∇F, which has been discussed in the previous section. 2) to show the existence of alternative optima, i.e., to show that the set of variables which yield the optimum is not unique, and 3) to demonstrate the capability of the synthesis method in design with fibrous composites. Four basic configurations of cylindrical shells are used for the Numerical Studies. The design data are presented in Tables 1–4. While the mechanical properties for the materials are presented in Table 5.

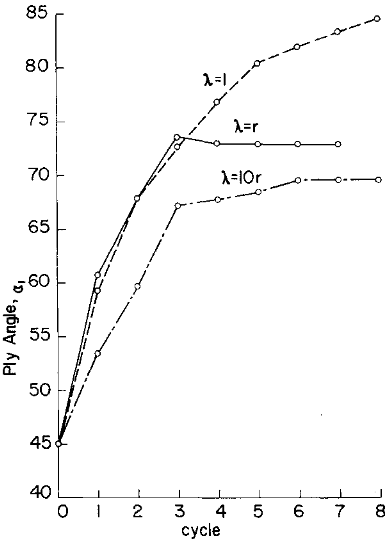
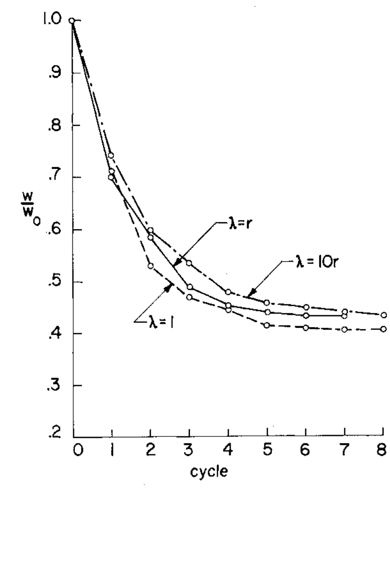


Fig. 3 Comparison of ply angle changes for various λ values, 2-ply skin shell under axial compression.

Table 1 is presented to illustrate the reasons and justifications for the modification of the gradient ∇F. A 2-ply skin cylindrical shell subjected to axial compression is used for the numerical illustrations. In the previous section, it was shown that if the modification factor λ is set equal to the constant multiplier r, i.e., $\lambda = r$, no modification is made on the gradient ∇F. If $\lambda = 10r$, the penalty on the partial derivatives of $B(\mathbf{x})$ with respect to ply angles α_i is ten times more severe than on other derivatives. Or if $\lambda = 1$, $\partial B/\partial \alpha_i$ are considered to be weighed numerically in the same scale as $\partial w/\partial x_m$. Numerical results for the three values of λ are summarized in Table 1. For these designs, the gross and panel buckling modes are suppressed to save computer time. However there is no loss in generality in the discussion which follows. All three designs start with the identical initial conditions. The skin buckling modes are critical for all three cases. In addition there is at least one active side constraint for each case. The result for $\lambda = 10r$ is clearly an alternative optimum for $\lambda = r$, i.e., the optimal values are the same but the sets of variables which produce the optimum are different. The ply angle changes are plotted against the numbers of cycles in Fig. 3. This plot shows that when $\lambda = r$ (Fiacco-McCormick method) the ply angle α_1 has little or no change after three cycles since the penalty term loses its effects. When $\lambda = 1$, α_1 can move in every cycle if it is profitable. In the first three cycles, the changes of α_1 are almost identical for $\lambda = r$ and $\lambda = 1$. After the third cycle, α_1 (for $\lambda = 1$) keeps changing but with diminishing magnitude. The final

Fig. 4 Weight reduction at each cycle for various λ values, 2-ply skin axial compression.



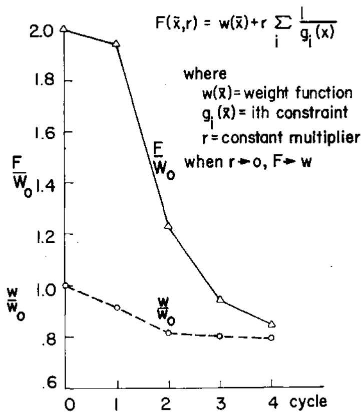


Fig. 5 Rates of change of F and W vs number of cycles, 4-ply skin shell under axial compression.

weight for $\lambda = 1$ is about 6.5% less than that for $\lambda = r$. And the ratio of load over critical load for skin buckling is 0.925 for $\lambda = 1$ in contrast with the value of 0.987 for $\lambda = r$. The weight reductions versus the number of cycles are shown in Fig. 4. For various values of λ, the curves seem to follow a similar path. The modification factor λ can also be used to fix the ply orientations at certain values. To accomplish this it is only necessary to set $\lambda = 0$ and to let each ply angle α_i be at the middle of its upper and lower bounds.

Unless stated to the contrary the value of λ will be set to one for the balance of the discussion. The primary purpose of the first three cases is to illustrate the basic characteristics of the synthesis scheme. Gross and panel buckling modes are included in the remainder of the design studies. The case shown in Table 2 is a four-ply skin shell subjected to axial compression. The stiffeners also have four-ply configurations and material failure in the longitudinal stiffeners are the critical failure modes. Figure 5 shows the rates of change of the weight vs the number of cycles. It should be noted that the minimization is actually carried out on the unconstrained function $F(\mathbf{x},r)$. The F value will decrease after each cycle but the w value may not. For this design the initial F value is twice that of the initial weight w_0. The design has terminated after four cycles, since further minimizations will only improve the minimum weight slightly. The initial and final ply orientations are shown in Fig. 6. This figure shows that the final configurations of the plies is almost equally spaced.

The cases presented in Table 3 have identical initial designs but different load conditions. This cylindrical shell configuration has a 3-ply skin and 2-ply stiffeners. The initial ply configuration in the skin is (+45°, −45°, +45°). Load condition 1 is an axial loading and the panel buckling is less critical than the initial design. Radial pressure is applied on the stiffened shell for the second load condition. The critical modes are the gross buckling and skin buckling. Load condition three is a torsional load. The skin material failure is critical throughout the design process. The design is trapped in a local minimum at the ply configuration (41°, −42°, 41°). However when the axial load and radial pressure are applied on the cylinder simultaneously with the torsional force, the design is able to escape from the local minimum but the skin material failure still dominates the design. This is a case of multiple load conditions and the results are summarized in Table 3. Since the material failure caused by the torsional load dominates the design throughout the synthesis process, the skin material failure mode will be neglected for the next example. The results thus obtained are much better in terms of a balanced design. The critical or near critical modes are gross buckling produced by torsional load, panel buckling produced by axial and torsional loads, and skin buckling produced by radial pressure.

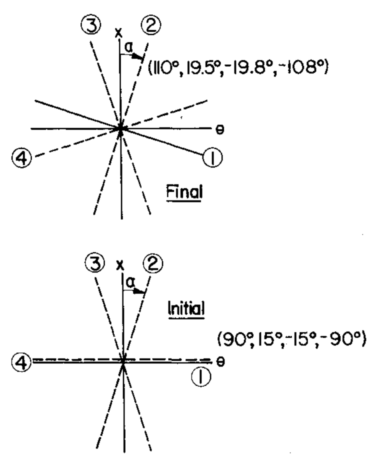


Fig. 6 Initial and final ply orientations in skin, 4-ply skin shell under axial compression.

With proper input, the program can be used to design isotropic cylindrical shells. Table 4 is a stiffened isotropic cylinder subjected to axial and radial loads. The critical modes for the final design are gross buckling, skin buckling, and skin material failure.

Throughout the investigation the gross buckling, panel buckling, skin buckling, and skin material failure seem to be more critical than the other failure modes. The failure modes of the stiffeners are important only if large lower bounds are placed on the dimensions of the stiffeners. For future work it is suggested to investigate the effects on the designs by placing stiffeners inside or outside the cylinder wall

Conclusions

The technique of structural synthesis, or automated optimal design of structural systems, has been extended to include the material parameters in the minimum weight design of stiffened fiber composite cylindrical shells. The fiber volume content in the design variable set serves two purposes: 1) vary the elastic constants of a unidirectional ply and 2) determine the thickness of the cylindrical shell. For merit functions independent of certain design variables the Fiacco-McCormick technique can be improved to give a faster convergence. These improvements have been obtained by a modification of the gradient of the unconstrained function $F(s,r)$.

Local optima exist in the problem in addition to alternative optima, i.e., the optimal value of the objective function for the problem is unique but the set of variables which yield the optimal value is not unique. In practice, the alternative optima may be a very useful property for a design engineer, since he may impose certain bounds on some variables and still achieve the optimal design. On the contrary, local optima must be checked out with several points.

The gradient modification matrix $[\lambda]$ can be used effectively to control the changes of the ply angles. For normal designs, $\lambda = 1$ gives adequate results. If the ply angles are to be fixed at certain positions, it is only necessary to set λ equal to zero and input the appropriate initial ply angles.

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General Instability of Eccentrically Stiffened Cylindrical Panels

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General instability of eccentrically stiffened thin cylindrical panels under the action of three types of applied loads is investigated. The three loads are uniform axial compression, uniform hoop compression or lateral pressure, and uniform shear. The analysis is based on a small-deflection theory for orthotropic shells which includes effects of stiffener eccentricity. Stiffener spacing is assumed to be sufficiently small (smeared technique), the stiffener geometry is taken to be uniform, and the skin-stiffener connection is assumed to be monolithic. The Galerkin procedure is employed to solve the buckling equations of the stiffened panels for the case of classical, simply supported boundaries. Critical loads were calculated from the resultant equations for two geometries: unstiffened isotropic panel and typical panel of the C-141 fuselage. Effects of stiffener eccentricity, panel aspect ratio, and the curvature parameter are shown in graphical form. When the initial curvature is set to zero, the results are applicable to flat rectangular simply supported plates. Finally, the C-5A Galaxy fuselage geometry was checked and shown to be safe for general instability failure.

Nomenclature

b = width of curved plate (along nonzero curvature)
D_{xx},D_{yy},D_{xy} = orthotropic flexural and twisting stiffnesses
e = stringer and ring eccentricities
$\bar{e}$ = nondimensionalized eccentricities [$= \pi^2 Re/L^2$]
E_{xx},E_{yy} = orthotropic extensional stiffnesses
G_{xy} = in-plane skin shear stiffness
k_x,k_y,k_s = applied load coefficients [see Eqs (3)]
L = length of curves plate (along zero curvature)
m = number of half-sine waves in x direction
M_x,M_y,M_{xy} = moment resultants
n = number of full-sine waves in y direction
N_x,N_y = stress resultants
$\bar{N}_x,\bar{N}_y,N_{xy}^0$ = membrane stress resultants
q = applied pressure (positive outward)
R = cylindrical panel radius (to skin midsurface)
w = normal displacement component (radial) of reference surface points
Z_t = curvature parameter [$= \{(1 - \nu^2)E_{xx}L^4/12R^2D_{yy}{}^{1/2}\}$]
α_{mn} = [see Eq. (8)]
λ_{ij},ρ_{ij} = extensional and flexural stiffness parameters [see Eq. (3)]
ν = Poisson's ratio
p = subscript referring to panel
r,st = subscripts referring to ring and stringer, respectively

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Introduction

General instability of eccentrically stiffened thin shells is one type of buckling and is defined as loss of stability in a mode in which all of the components of the stiffened shell (i.e., the stiffeners and skin) participate and the deformation extends over the entire surface of the shell. This type of buckling of eccentrically stiffened complete thin cylindrical shells has been extensively studied by many authors since the 1930's. (For a fairly complete bibliography, see Refs. 1 and 2.)

Since the complete cylinder results are not directly applicable to the stiffened partial cylindrical panel, general instability of such structural elements is investigated, with special attention to the effects of stiffener eccentricity, curvature, and panel aspect ratio, under the action of three types of applied loads. These three types of loads are uniform axial compression, uniform hoop compression or lateral pressure, and uniform shear.

Stability studies of curved cylindrical panels of isotropic geometry under pressure, shear, and axial compression have been reported by Rafel,[3,4] Rafel and Sandlin,[5] Schildcraut and Stein,[6] and Batdorf et al.[7,8] the effect of a single stiffening member on the stability of curved cylindrical panels has been reported by Stein and Yaeger[9] and Batdorf and Schildcrout.[10] For a more complete bibliography and a comprehensive discussion of these problems, see Refs. 11 and 12.

In the early 1950's, a theory was developed by Stein and Mayers[13] to analyze curved plates of sandwich construction. The same authors[14] used their theory to predict buckling of